This lecture first introduces the concepts of correspondences and their continuity, and then discuss two important results, Kakutani’s fixed point theorem and Berge’s theorem of maximum. You may refer to FMEA Chapter 14.1, 14.2, and 14.4.

1 Definitions

Definition 1.1. A correspondence $F$ from $X$ to $Y$ is a set-valued function that associates every element in $X$ a subset of $Y$.

$$F : X \rightrightarrows Y$$

$$x \mapsto F(x)$$

The set $X$ is called the domain of the correspondence $F$, and $Y$ is called the codomain of $F$. $F(x)$ is called the image of point $x \in X$.

You may consider the concept of correspondence as a generalization of functions, in the sense that $F(x)$ is a set in $Y$ instead of an element in $Y$. Clearly, a single-valued correspondence $F : X \rightrightarrows Y$ can be viewed as a function from $X$ to $Y$.

Listed below are some terminologies that we use to describe the properties of correspondences.
Definition 1.2. A correspondence $F : X \rightrightarrows Y$ is said to be XXX-valued at $x_0 \in X$ iff $F(x_0)$ is a XXX set. If $F$ is XXX-valued at all $x_0 \in X$, we say $F$ is XXX-valued.

These "XXX" can be

1. non-empty
2. single (singleton)
3. open
4. closed
5. compact
6. convex

Notice that the 3 - 5 above requires $Y$ to be a metric space $(Y, d_Y)$, and 6 requires $Y$ to be a (real) vector space $(Y, +, \cdot)$.

1.1 Upper Hemi-contuity

Similar to functions, it is possible to talk about continuity of a correspondence if its domain and codomain are both metric spaces. However, there are two distinct notions of continuity for correspondences, known as upper hemi-continuity and lower hemi-continuity, and they capture different aspects of continuity of a correspondence. Let’s first look at upper hemi-continuity.

Definition 1.3. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be upper hemi-continuous (uhc) at $x_0 \in X$ iff $\forall$ open set $U$ in $(Y, d_Y)$ s.t. $F(x_0) \subset U$, $\exists \delta > 0$ s.t. $F(B_\delta(x_0)) \subset U$.

The correspondence $F : X \rightrightarrows Y$ is said to be upper hemi-continuous (uhc) iff it is upper hemi-continuous at $x_0$ for all $x_0 \in X$.

The definition requires that whenever the open set $U$ covers the entire image of the point $x_0$, then it must also entirely cover all nearby images. What is not allowed by uhc at $x_0$ is sudden appearance of large chunk of image when $x$ deviates from $x_0$.

For example, consider the correspondence $F_1 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_1(x) := \begin{cases} 
\{0\}, & \text{if } x \leq 0 \\
[-1, 1], & \text{if } x > 0 
\end{cases}$$

Clearly $F_1$ fails to be uhc at 0, because if we let $U := (-1/2, 1/2)$, whenever $x$ moves away a little from 0 to the right, the image $F_1(x)$ becomes $[-1, 1]$, which is not covered by $U$. The problem of this correspondence at 0 is that many new points suddenly appear when $x$ deviates from 0 to the right, and this is a violation of uhc. Therefore, uhc can be intuitively interpreted as “no sudden appearance of large chunk of image when deviating from a point”.

Consider a slightly different correspondence $F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_2(x) := \begin{cases} 
\{0\}, & \text{if } x < 0 \\
[-1, 1], & \text{if } x \geq 0 
\end{cases}$$

In $\mathbb{R}^n$, we use the Euclidean metric $d_2$ by default.
The image of \( F_2 \) at 0 is \([-1, 1]\), and so there is no sudden appearance of image when deviating from 0. Therefore, \( F_2 \) is uhc at 0. Clearly, \( F_2 \) is also uhc at all other points in \( \mathbb{R} \), and so \( F_2 \) is uhc.

Uhc does not allow sudden appearance of image when deviating from a point, but it allows ”smooth changes” in the image when deviating from a point, if the correspondence is closed-valued at this point. For example, consider the correspondence \( F_3 : \mathbb{R} \Rightarrow \mathbb{R} \) defined as

\[
F_3(x) := [x, x + 1]
\]

for any \( x \in \mathbb{R} \). Under \( F_3 \), the image \( F_3(x) = [x, x + 1] \) changes ”smoothly” when \( x \) changes, and it can be shown that \( F_3 \) is uhc.

**Claim 1.4.** The correspondence \( F_3 : \mathbb{R} \Rightarrow \mathbb{R} \) defined above is uhc.

**Proof.** Take any \( x_0 \in \mathbb{R} \). WTS: \( F_3 \) is uhc at \( x_0 \).

Take any open set \( U \supset [x_0, x_0 + 1]\). WTS: \( \exists \delta > 0 \) s.t. \( U \supset F(x) \) for any \( x \in (x_0 - \delta, x_0 + \delta) \).

Because \( x_0 \) and \( x_0 + 1 \) are in the open set \( U \), they are interior points of \( U \), and so \( \exists \delta > 0 \) s.t.

\[
(x_0 - \delta, x_0 + \delta) \subset U \\
(x_0 + 1 - \delta, x_0 + 1 + \delta) \subset U
\]

Therefore, we have \( (x_0 - \delta, x_0 + 1 + \delta) \subset U \).

For any \( x \in (x_0 - \delta, x_0 + \delta) \), we have

\[
F(x) = [x, x + 1] \subset (x_0 - \delta, x_0 + 1 + \delta) \subset U
\]

However, when the correspondence is not closed-valued, then even smooth changes in the image may violate uhc. For example, consider a slightly different correspondence \( F_4 : \mathbb{R} \Rightarrow \mathbb{R} \) defined as

\[
F_4(x) := (x, x + 1)
\]

It can be shown that it is not uhc at any point in \( \mathbb{R} \). To see this, for each \( x_0 \in \mathbb{R} \), let \( U := F_4(x_0) := (x_0, x_0 + 1) \), and \( U \) cannot cover \( F(x) \) as long as \( x \neq x_0 \).

In applications, however, we almost always work with closed-valued correspondences, in which case uhc allows smooth changes, but does not allow sudden appearance of image.

For single-valued correspondences, uhc is equivalent to continuity of functions.

**Proposition 1.5.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. Consider a single-valued correspondence \( F : X \Rightarrow Y \). Define \( f : X \rightarrow Y \) as \( f(x) := y \) s.t. \( y \in F(x) \). Then \( F \) is uhc at \( x_0 \in X \) iff \( f \) is continuous at \( x_0 \).

This proof is straightforward, and is left as an exercise.

For compact-valued correspondences, there is a **sequential definition of uhc**, which is formulated in the following proposition\(^3\).

**Proposition 1.6.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. Consider a correspondence \( F : X \Rightarrow Y \), and let \( x_0 \in X \). Then the following two statements are equivalent:

1. \( F \) is compact-valued at \( x_0 \), and \( F \) is uhc at \( x_0 \).
2. For any sequence \( (x_n) \) in \( X \) convergent to \( x_0 \), any sequence \( (y_n) \) s.t. \( y_n \in F(x_n) \) for each \( n \in \mathbb{N} \), there exists a subsequence \( (y_{n_k}) \) convergent to some \( y_0 \in F(x_0) \).

\(^3\)This is the definition of uhc in the book by SLP, who only study compact-valued correspondences.
Therefore, we have a sequence of \((y_n)\) that converges to some \(y_0 \in F(x_0)\). For each \(k \in \mathbb{N}\), consider the set

\[ U_k := \bigcup_{y \in F(x_0)} B_{1/k}(y) \]

Because arbitrary union of opens is still open, we know that \(U_k\) is an open set. By construction \(U_k \supset F(x_0)\), and so by uhc of \(F\) at \(x_0\), there exists \(\delta_k > 0\) s.t. \(F(B_{\delta_k}(x_0)) \subset U_k\). Because \(x_n \to x_0\), there exists \(N_k\) s.t. \(x_n \in B_{\delta_k}(x_0)\), and thus \(y_n \in U_k\) for any \(n > N_k\).

Therefore, we can find a subsequence \((y_{nk})\) s.t. \(y_{nk} \in U_k\) for each \(k \in \mathbb{N}\). By construction of \(U_k\), for each \(k\), there exists \(z_k \in F(x_0)\) s.t. \(d_Y(y_{nk}, z_k) < 1/k\). Because \(F\) is compact-valued at \(x_0\), we know that \(F(x_0)\) is compact in \((Y, d_Y)\). So there exists a subsequence \((z_{kl})\) convergent to some \(y_0 \in F(x_0)\). So we have \(d_Y(z_{kl}, y_0) \to 0\), and

\[
0 \leq d_Y(y_{nk}, y_0) \leq d_Y(y_{nk}, z_{kl}) + d_Y(z_{kl}, y_0) < \frac{1}{kl} + d_Y(z_{kl}, y_0) \to 0 + 0 = 0
\]

Therefore, we have \(d_Y(y_{nk}, y_0) \to 0\), which means \(y_{nk} \to y_0\). Therefore, we have found a subsequence of \((y_n)\) that converges to some point in \(F(x_0)\).

\[ (1) \iff (2): \]

(a) WTS: \(F\) is compact-valued at \(x_0\).

Take any sequence \((y_n)\) in \(F(x_0)\). WTS: There exists a subsequence \((y_{nk})\) convergent to some \(y_0 \in F(x_0)\).

Let \(x_n = x_0\) for all \(n \in \mathbb{N}\). Then we have \(x_n \to x_0\) and \(y_n \in F(x_n)\) for each \(n \in \mathbb{N}\). By assumption, there exists a subsequence \((y_{nk})\) convergent to some \(y_0 \in F(x_0)\).

(b) WTS: \(F\) is uhc at \(x_0\).

Suppose that \(F\) is not uhc at \(x_0\). Then \(\exists U\) open in \((X, d_X)\) s.t. \(U \supset F(x_0)\), but \(\forall \delta > 0\) we have \(U \not\supset F(B_\delta(x_0))\). Then for any \(n \in \mathbb{N}\), we have \(U \not\supset F(B_{1/n}(x_0))\), i.e. there exists \(x_n \in B_{1/n}(x_0)\) and \(y_n \in F(x_n)\) s.t. \(y_n \notin U\). Because \(x_n \to x_0\), by assumption there exists a subsequence \((y_{nk})\) convergent to some \(y_0 \in F(x_0)\).

Because \((y_{nk})\) is in \(Y \setminus U\), which is closed in \((Y, d_Y)\), we have \(y_0 \in Y \setminus U\), and so \(y_0 \notin F(x_0)\). Contradiction.

Without compact-valuedness, uhc alone does not imply property (2) in the proposition above. For example, consider \(F_5 : \mathbb{R} \to \mathbb{R}\) defined as

\[ F_5(x) = (0, 1) \]

for any \(x \in \mathbb{R}\). Clearly, \(F_5\) is uhc everywhere, but it does not satisfy property (2) at any \(x_0 \in \mathbb{R}\), since compact-valuedness is necessary for property (2).

### 1.2 Closed Graph Property

There is a concept, called *closed graph property*, that is closely related to uhc.
**Definition 1.7.** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. The correspondence \(F : X \Rightarrow Y\) is said to have **closed graph property (cgp)** at \(x_0 \in X\) iff \(\forall\) sequence \((x_n)\) in \(X\) convergent to \(x_0\), 
\(y_n \in F(x_n)\) for each \(n \in \mathbb{N}\), and \(y_n \to y_0 \in Y\), we have \(y_0 \in F(x_0)\).

The correspondence \(F : X \Rightarrow Y\) is said to have **closed graph property (cgp)** iff it has closed graph property at \(x_0\) for all \(x_0 \in X\).

Clearly, cgp implies closed-valuedness.

**Claim 1.8.** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. If the correspondence \(F : X \Rightarrow Y\) is cgp at \(x_0 \in X\), then it is closed-valued at \(x_0\).

**Proof.** Take any sequence \((y_n)\) in \(F(x_0)\) convergent to \(y_0 \in Y\). WTS: \(y_0 \in F(x_0)\).

Let \(x_n = x_0\) for all \(n \in \mathbb{N}\), then we have \(x_n \to x_0\), \(y_n \in F(x_n)\) for each \(n \in \mathbb{N}\), and \(y_n \to y_0 \in Y\). By cgp, we have \(y_0 \in F(x_0)\). □

The **graph**\(^\dagger\) of a correspondence \(F : X \Rightarrow Y\) is defined as

\[
Gr(F) := \{ (x,y) \in X \times Y : y \in F(x) \}
\]

For a correspondence \(F : X \Rightarrow Y\), where \((X,d_X)\) and \((Y,d_Y)\) are metric spaces, the name of the property "closed graph property" comes from the fact that \(F\) has cgp (everywhere in \(X\)) iff its graph is closed in \((X \times Y,d_{X \times Y})\), where the metric for the product space is defined as

\[
d_{X \times Y}((x,y),(x',y')) := \sqrt{[d_X(x,x')]^2 + [d_Y(y,y')]^2}
\]

for any \((x,y), (x',y') \in X \times Y\).

It can be shown that \(d_{X \times Y}\) as defined above is a valid metric for \(X \times Y\). Also, we can show that \((x_n,y_n) \to (x_0,y_0)\) in \((X \times Y,d_{X \times Y})\) iff \(x_n \to x_0\) in \((X,d_X)\) and \(y_n \to y_0\) in \((Y,d_Y)\), and this is left as an exercise.

**Claim 1.9.** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Then a correspondence \(F : X \Rightarrow Y\) has cgp iff \(Gr(F)\) is closed in \((X \times Y,d_{X \times Y})\).

**Proof.** ⇒:

Take any \(((x_n,y_n))\) in \(Gr(F)\) that is convergent to \((x_0,y_0) \in X \times Y\). WTS: \((x_0,y_0) \in Gr(F)\).

Because \((x_n,y_n) \to (x_0,y_0)\), we have \(x_n \to x_0\) and \(y_n \to y_0\). Because \((x_n,y_n) \in Gr(F)\) for all \(n\), we have \(y_n \in F(x_n)\) for all \(n\). Because \(F\) has cgp, we know that \(F\) has cgp at \(x_0\), and so \(y_0 \in F(x_0)\), which implies \((x_0,y_0) \in Gr(F)\).

⇐:

Take any \(x_0 \in X\). WTS: \(F\) has cgp at \(x_0\).

Take any \((x_n)\) in \(X\) convergent to \(x_0\), \(y_n \in F(x_n)\) for each \(n \in \mathbb{N}\), and \(y_n \to y_0 \in Y\). WTS: \(y_0 \in F(x_0)\).

Because \(x_n \to x_0\) and \(y_n \to y_0\), we have \((x_n,y_n) \to (x_0,y_0)\) in \((X \times Y,d_{X \times Y})\). Because \(y_n \in F(x_n)\) for each \(n\), we have \((x_n,y_n) \in Gr(F)\) for each \(n\). Because \(Gr(F)\) is closed in \((X \times Y,d_{X \times Y})\), we have \((x_0,y_0) \in Gr(F)\). □

Closed graph property is closely related to uhc, and their relation is formulated by the following two propositions.

\(^\dagger\)This is in fact a redundant definition since \(Gr(F) = F\), if we view \(F\) as a relation from \(X \times Y\).
Proposition 1.10. Let \((X, d_X) \) and \((Y, d_Y) \) be metric spaces. If a correspondence \(F : X \rightrightarrows Y \) is uhc at \(x_0 \in X \), and is closed-valued at \(x_0 \), then \(F \) has cgp at \(x_0 \).

Proof. Take any sequence \((x_n) \) in \(X \) convergent to \(x_0 \), and \(y_n \in F(x_n) \) for each \(n \in \mathbb{N} \), and \(y_n \to y_0 \in Y \).

WTS: \(y_0 \in F(x_0) \).

Suppose \(y_0 \notin F(x_0) \), i.e. \(y_0 \notin Y \setminus F(x_0) \). Because \(F \) is closed-valued at \(x_0 \), \(Y \setminus F(x_0) \) is open in \((Y, d_Y) \), and so \(\exists \varepsilon > 0 \) s.t. \(B_{2\varepsilon}(y_0) \subset Y \setminus F(x_0) \). And the "closed ball"

\[
\bar{B}_\varepsilon(y_0) := \{ y \in Y : d_Y(y, y_0) \leq \varepsilon \}
\]

is contained in \(B_{2\varepsilon}(y_0) \) and therefore in \(Y \setminus F(x_0) \), and therefore \(F(x_0) \subset Y \setminus \bar{B}_\varepsilon(y_0) \). It can be shown that a closed ball is a closed set (exercise), and \(F(x_0) \) is covered by the open set \(Y \setminus \bar{B}_\varepsilon(y_0) \).

By uhc of \(F \) at \(x_0 \), \(\exists \delta > 0 \) s.t. \(F(B_\delta(x_0)) \subset Y \setminus \bar{B}_\varepsilon(y_0) \).

Because \(x_n \to x_0 \) and \(y_n \to y_0 \), there exists \(n \) s.t. \(x_\hat{n} \in B_\delta(x_0) \) and \(y_\hat{n} \in \bar{B}_\varepsilon(y_0) \). However, because \(F(B_\delta(x_0)) \subset Y \setminus \bar{B}_\varepsilon(y_0) \), we have \(y_\hat{n} \in F(x_\hat{n}) \subset F(B_\delta(x_0)) \subset Y \setminus \bar{B}_\varepsilon(y_0) \), which contradicts \(y_\hat{n} \in \bar{B}_\varepsilon(y_0) \).

The result above states that uhc implies cgp if we have closed-valuedness. Without closed-valuedness, this implication does not hold since a uhc correspondence may not have closed-valuedness.

A correspondence \(F : X \rightrightarrows Y \), where \((X, d_X) \) and \((Y, d_Y) \) are metric spaces, is said to be locally bounded at \(x_0 \) iff \(\exists \delta > 0 \) and a compact set \(K \) in \((Y, d_Y) \) s.t. \(F(B_\delta(x_0)) \subset K \).

The next proposition works in the other direction.

Proposition 1.11. Let \((X, d_X) \) and \((Y, d_Y) \) be metric spaces. If a correspondence \(F : X \rightrightarrows Y \) has cgp at \(x_0 \in X \), and \(F \) is locally bounded at \(x_0 \), then \(F \) is uhc at \(x_0 \).

The proof of this proposition is similar to the proof of Proposition 1.6, part (b) of the direction "(1) \(\Rightarrow\) (2)".

Proof. Suppose that \(F \) is not uhc at \(x_0 \). Then \(\exists U \) open in \((Y, d_Y) \) s.t. \(F(x_0) \subset U \), but \(\forall \delta > 0 \) we have \(F(B_\delta(x_0)) \subset U \). Then for any \(n \in \mathbb{N} \), we have \(F(B_{1/n}(x_0)) \not\subset U \), i.e. there exists \(x_n \in B_{1/n}(x_0) \) and \(y_n \in F(x_n) \) s.t. \(y_n \not\in U \). By assumption there exists \(\hat{\delta} > 0 \) and compact set \(K \) in \((Y, d_Y) \) s.t. \(F(B_{\hat{\delta}}(x_0)) \subset K \). By construction, we have \(x_n \to x_0 \), and so \(\exists N \) s.t. \(x_n \in B_{\hat{\delta}}(x_0) \) and \(y_n \in K \) for any \(n > N \).

By sequential compactness of \(K \), there exists a subsequence \((y_{n_k}) \) of \((y_n)_{n>N} \) convergent to some \(y_0 \in K \). Because the subsequence \((y_{n_k}) \subset Y \setminus U \), which is closed, we have \(y_0 \in Y \setminus U \). However, because \(F \) has cgp at \(x_0 \), and \(x_{n_k} \to x_0 \), \(y_{n_k} \in F(x_{n_k}) \), \(y_{n_k} \to y_0 \), we have \(y_0 \in F(x_0) \subset U \).

Contradiction.

The result above states that cgp implies uhc if we have local boundedness. Without local boundedness, cgp does not imply uhc. For example, consider \(F_6 : \mathbb{R} \rightrightarrows [0, 1] \) defined as

\[
F_6(x) = \begin{cases} 
\{x^2\}, & x < 0 \\
\{0\}, & x \geq 0
\end{cases}
\]

which is clearly not uhc at 0. However, \(F_6 \) has cgp at 0. Notice that 1 is not in the codomain, and so when \(x_n \) converges to 0 from the negative real line, \(y_n \in F(x_n) \) does not converge. This is not a violation of the proposition above, because \(F_6 \) is not locally bounded at 0. Notice again that 1 is
not in the codomain, and so we cannot find a compact set $K$ in $(0, 1), d_2$ to bound all images of points nearby 0.

Another example is $F_7 : \mathbb{R} \Rightarrow \mathbb{R}$ defined as

$$F_6(x) = \begin{cases} 
\{1/x\}, & x \neq 0 \\
\{0\}, & x = 0 
\end{cases}$$

As a consequence of the two propositions above, under closed-valuedness and local boundedness, uhc and cgp are equivalent.

### 1.3 Lower Hemi-continuity

Now let’s define lower hemi-continuity.

**Definition 1.12.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. The correspondence $F : X \Rightarrow Y$ is said to be lower hemi-continuous (lhc) at $x_0 \in X$ iff $\forall$ open set $U$ in $(Y, d_Y)$ s.t. $F(x_0) \cap U \neq \emptyset$, $\exists \delta > 0$ s.t. $F(x) \cap U \neq \emptyset$ for any $x \in B_\delta(x_0)$.

The correspondence $F : X \Rightarrow Y$ is said to be lower hemi-continuous (lhc) iff it is lower hemi-continuous at $x_0$ for all $x_0 \in X$.

The definition requires that whenever the open set $U$ covers a part of the image of the point $x_0$, then it must also cover a part of all nearby images. What is not allowed by lhc at $x_0$ is sudden disappearance of large chunk of image when $x$ deviates from $x_0$.

For example, consider the correspondence $F_2 : \mathbb{R} \Rightarrow \mathbb{R}$

$$F_2(x) := \begin{cases} 
\{0\}, & x < 0 \\
[-1, 1], & x \geq 0 
\end{cases}$$

as previously defined. Clearly $F_2$ fails to be lhc at 0, because if we let $U := (1/2, 3/2)$, whenever $x$ moves away a little from 0 to the left, the image $F_2(x)$ becomes $\{0\}$, which does not share an intersection with $U$. The problem of this correspondence at 0 is that many points suddenly disappear when $x$ deviates from 0 to the left, and this is a violation lhc. Therefore, lhc can be intuitively interpreted as ”no sudden disappearance of large chunk of image when deviating from a point”.

Consider the slightly different correspondence $F_1 : \mathbb{R} \Rightarrow \mathbb{R}$

$$F_1(x) := \begin{cases} 
\{0\}, & x \leq 0 \\
[-1, 1], & x > 0 
\end{cases}$$

as previously defined. The image of $F_1$ at 0 is $\{0\}$, and so there is no sudden disappearance of image when deviating from 0. Therefore, $F_1$ is lhc at 0. Clearly, $F_1$ is also lhc at all other points in $\mathbb{R}$, and so $F_1$ is lhc.

Lhc does not allow sudden disappearance of image when deviating from a point, but it allows ”smooth changes” in the image when deviating from a point. For example, consider the correspondence $F_3 : \mathbb{R} \Rightarrow \mathbb{R}$

$$F_3(x) := [x, x + 1]$$

for any $x \in \mathbb{R}$ as previously defined. Under $F_3$, the image $F_3(x) = [x, x + 1]$ changes ”smoothly” when $x$ changes, and it can be shown that $F_3$ is lhc.

**Claim 1.13.** The correspondence $F_3 : \mathbb{R} \Rightarrow \mathbb{R}$ defined above is lhc.
Proof. Take any \( x_0 \in \mathbb{R} \). WTS: \( F_3 \) is lhc at \( x_0 \).

Take any open set \( U \) s.t. \( [x_0, x_0 + 1] \cap U \neq \emptyset \).

WTS: \( \exists \delta > 0 \) s.t. \( [x, x + 1] \cap U \neq \emptyset \) for any \( x \in (x_0 - \delta, x_0 + \delta) \).

Let \( \hat{x} \in [x_0, x_0 + 1] \cap U \). Because \( U \) is open, there exists \( \delta > 0 \) s.t. \( (\hat{x} - \delta, \hat{x} + \delta) \subset U \).

Take any \( x \in (x_0 - \delta, x_0 + \delta) \). By construction, we have \( x - x_0 \in (-\delta, \delta) \), and so

\[
\hat{x} + (x - x_0) \in (\hat{x} - \delta, \hat{x} + \delta) \subset U
\]

Because \( \hat{x} \in [x_0, x_0 + 1] \), we have

\[
\hat{x} + (x - x_0) \in [x_0 + (x - x_0), x_0 + (x - x_0) + 1] = [x, x + 1]
\]

Therefore, we have \( \hat{x} + (x - x_0) \in [x, x + 1] \cap U \), and so \( [x, x + 1] \cap U \neq \emptyset \).

Lhc allows smooth changes in the image, regardless of whether the correspondence is closed valued. If we consider a slightly different correspondence \( F_4 : \mathbb{R} \Rightarrow \mathbb{R} \) defined as

\[
F_4 (x) := (x, x + 1)
\]

for any \( x \in \mathbb{R} \), a slightly modification of the proof above can show that \( F_4 \) is also lhc.

For single-valued correspondences, lhc is equivalent to continuity of functions.

**Proposition 1.14.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. Consider a single-valued correspondence \( F : X \Rightarrow Y \). Define \( f : X \to Y \) as \( f(x) := y \) s.t. \( y \in F(x) \). Then \( F \) is lhc at \( x_0 \in X \) iff \( f \) is continuous at \( x_0 \).

This proof is straightforward, and is left as an exercise.

The following proposition provides the **sequential definition of lhc.**

**Proposition 1.15.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. A correspondence \( F : X \Rightarrow Y \) is lhc at \( x_0 \in X \) iff for any \( y_0 \in F(x_0) \) and sequence \( (x_n) \) in \( X \) convergent to \( x_0 \), there exists \( N \in \mathbb{N} \) and \( y_n \in F(x_n) \) for any \( n > N \) s.t. the sequence \( (y_n)_{n>N} \) converges to \( y_0 \).

In the proposition above, we start to construct the sequence \( (y_n) \) starting from \( n = N + 1 \), because \( F(x_n) \) may be empty for small \( n \)’s.

**Proof.** \( \Rightarrow \):

Take any \( y_0 \in F(x_0) \) and sequence \( (x_n) \) in \( X \) convergent to \( x_0 \).

WTS: \( \exists N \in \mathbb{N} \) and \( y_n \in F(x_n) \) for any \( n > N \) s.t. the sequence \( (y_n)_{n>N} \) converges to \( y_0 \).

For each \( k \in \mathbb{N} \), we have \( y_0 \in F(x_0) \cap B_{1/k} (y_0) \), and so \( F(x_0) \cap B_{1/k} (y_0) \neq \emptyset \). By lhc, \( \exists \delta_k > 0 \) s.t. for any \( x \in B_{\delta_k} (x_0) \), we have \( F(x) \cap B_{1/k} (y_0) \neq \emptyset \).

Because \( x_n \to x \), \( \exists N \in \mathbb{N} \) s.t. \( x_n \in B_{\delta_k} (x_0) \) for any \( n > N \).

For each \( n > N \), arbitrarily take

\[
y_n \in \bigcap_{k \in \mathbb{N} : x_n \in B_{\delta_k} (x_0)} F(x_n) \cap B_{1/k} (y_0)
\]

This is possible because \( F(x_n) \cap B_{1/k} (y_0) \neq \emptyset \) whenever \( x_n \in B_{\delta_k} (x_0) \).

Now I want to show that \( (y_n)_{n>N} \) converges to \( y_0 \).
Take any \( \varepsilon > 0 \). \( \exists K \) s.t. \( 1/k < \varepsilon \) for any \( k > K \). Because \( x_n \to x_0, \exists \hat{N} > N \) s.t. \( x_n \in B_{\delta_K}(x_0) \) for any \( n > \hat{N} \). Therefore for any \( n > \hat{N} \), we have \( x_n \in B_{\delta_K}(x_0) \), which implies \( y_n \in B_{1/K}(y_0) \), which in turn implies \( y_n \in B_{\varepsilon}(y_0) \).

\[
\iff
\]
Suppose, by contradiction, that \( \exists \) open set \( U \) in \((Y, d_Y)\) s.t. \( F(x_0) \cap U \neq \emptyset \), but \( \forall \delta > 0, \exists x \in B_\delta(x_0) \) s.t. \( F(x) \cap U = \emptyset \). This implies that for any \( n \in \mathbb{N} \), \( \exists x_n \in B_{1/n}(x_0) \) s.t. \( F(x_n) \cap U = \emptyset \), i.e. \( F(x_n) \subseteq Y \setminus U \).

By construction, we have \( x_n \to x_0 \). Arbitrarily take \( y_0 \in F(x_0) \cap U \), and by assumption there exists \( N \in \mathbb{N} \) and \( y_n \in F(x_n) \) for any \( n > N \) s.t. the sequence \((y_n)_{n>N}\) converges to \( y_0 \). Because \( y_n \in F(x_n) \subseteq Y \setminus U \) for any \( n > N \), and \( Y \setminus U \) is closed in \((Y, d_Y)\) since \( U \) is open, we have \( y_0 \in Y \setminus U \). This contradicts the construction of \( y_0 \).

As we have discussed, uhc for closed-valued correspondences means no sudden appearance of image when deviating from a point, while uhc means no sudden disappearance of image when deviating from a point. Therefore, we might expect a relation between \( \text{uhc} \) and \( \text{lhcf} \). In fact, we have one direction, but not the other.

For a correspondence \( F : X \rightrightarrows Y \), let’s define its complement \( F^c : X \rightrightarrows Y \) as

\[
F^c(x) := Y \setminus F(x)
\]
for any \( x \in X \). Again, this is a redundant definition if we realize that \( F \) is essentially a subset of \( X \times Y \).

**Proposition 1.16.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, and consider a correspondence \( F : X \rightrightarrows Y \). If \( F^c \) is uhc at \( x_0 \in X \), then \( F \) is lhcf at \( x_0 \).

The proof is left as an exercise.

However, \( F^c \) being lhcf does not imply \( F \) being uhc, even if we further assume \( F \) to be compact-valued. For example, consider the correspondence \( F_7 : \mathbb{R} \rightrightarrows \mathbb{R} \) defined as:

\[
F_8(x) := \{\begin{cases} \{0\}, & \text{if } x < 0 \\ \{1\}, & \text{if } x \geq 0 \end{cases}
\]

Clearly \( F \) is compact-valued, and \( F(x) \) is not uhc at \( 0 \). However, \( F^c \) is lhcf at all \( x_0 \in \mathbb{R} \).

Finally, a correspondence is said to be continuous iff it is both uhc and lhcf.

**Definition 1.17.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. The correspondence \( F : X \rightrightarrows Y \) is said to be **continuous at** \( x_0 \in X \) iff \( F \) is both uhc and lhcf at \( x_0 \). The correspondence \( F \) is said to be **continuous** iff \( F \) is continuous at \( x_0 \) for all \( x_0 \in X \).

## 2 Kakutani’s Fixed Point Theorem

**Definition 2.1.** A correspondence \( F \) from \( X \) to \( X \) itself is called a **self-correspondence**.

For a self-correspondence \( F : X \rightrightarrows X \), a point \( x^* \in X \) is called a **fixed point** of \( F \) iff \( x^* \in F(x^*) \).

When the self-correspondence \( F \) is single-valued, clearly \( x^* \in X \) is a fixed point of \( F \) iff \( F(x^*) = \{x^*\} \), which is consistent with notion of fixed points for functions. Therefore, the definition above can be considered as a generalization of the notion of fixed points to correspondences.
**Theorem 2.2** (Kakutani’s Fixed Point). Let $X$ be a nonempty, compact, and convex set in $\mathbb{R}^n$. If the self-correspondence $F : X \rightrightarrows X$ is nonempty-valued, compact-valued, convex-valued, and uhc, then there exists a fixed point $x^* \in X$ of $F$.

In the theorem above, compactness is w.r.t. the metric space $(\mathbb{R}^n, d_2)$, and convexity is w.r.t. the vector space $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$, where $+$ and $\cdot$ are the usually defined vector addition and scalar multiplication for real vectors.

If $F$ is single-valued, then nonempty-valuedness, compact-valuedness, and convex-valuedness of $F$ holds trivially, and uhc reduces to the continuity of functions. So the theorem above reduces to Brouwer’s fixed point theorem. Therefore, Kakutani’s fixed point theorem should be viewed as a generalization of Brouwer’s fixed point theorem.

Because the codomain $X$ of $F$ is compact in the theorem, compact-valuedness is equivalent to closed-valuedness, and so we can replace the compact-valuedness assumption by closed-valuedness.

Again because the codomain $X$ is compact, (compact-valuedness + uhc) is equivalent to cgp. To see this, the direction ”$\Rightarrow$” is given by Proposition 1.10, and the other direction ”$\Leftarrow$” is given by Proposition 1.11, since local boundedness holds trivially. Therefore we have the following corollary.

**Corollary 2.3.** Let $X$ be a nonempty, compact, and convex set in $\mathbb{R}^n$. If the self-correspondence $F : X \rightrightarrows X$ is nonempty-valued, convex-valued, and has cgp, then there exists a fixed point $x^* \in X$ of $F$.

Kakutani’s fixed point theorem plays the central role in the proof of the existence of Walrasian equilibrium in general equilibrium theory, and also in the proof of the existence of Nash equilibrium in non-cooperative game theory.

### 3 Berge’s Theorem of Maximum

**Theorem 3.1.** (Berge’s Theorem of Maximum) Let $(X,d_X)$ and $(A,d_A)$ be metric spaces. Let $f : X \times A \to \mathbb{R}$ be a continuous function w.r.t. the metric $d_{X \times A}$. Let $\alpha_0 \in A$, and suppose that the correspondence $D : A \rightrightarrows X$ is nonempty-valued at $\alpha_0$, compact-valued at $\alpha_0$, and continuous at $\alpha_0$.

Define the correspondence $X^* : A \rightrightarrows X$ as

$$X^*(\alpha) := \text{arg max}_{x \in X} \{ f(x, \alpha) : x \in D(\alpha) \}$$

for any $\alpha \in A$.

Let $\hat{A} := \{ \alpha \in A : X^*(\alpha) \neq \emptyset \}$, and define the function $f^* : \hat{A} \to \mathbb{R}$ as

$$f^*(\alpha) = \max_{x \in X} \{ f(x, \alpha) : x \in D(\alpha) \}$$

Then $X^*$ is nonempty-valued at $\alpha_0$, compact-valued at $\alpha_0$, and uhc at $\alpha_0$, and $f^*$ is continuous at $\alpha_0$.

In the theorem above, the maximization problem we are looking at is a parameterized problem

$$\max_{x \in X} f(x, \alpha) \text{ s.t. } x \in D(\alpha)$$

where both the objective function $f$ and the constraint set $D$ depend on the parameter $\alpha$. The theorem states that if the objective function $f$ is continuous, and the constraint set $D$ is nonempty and compact-valued, and is both uhc and lhc in the parameter $\alpha$, then the set of maximizers $X^*$ is compact and uhc in $\alpha$, and the maximum value $f^*$ is also continuous in $\alpha$. 

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Proof. Let’s prove the theorem in three steps:

**Step 1: $X^*$ is nonempty-valued at $\alpha_0$**

Because $f: X \times A \to \mathbb{R}$ is continuous w.r.t. $d_{X \times A}$, clearly the function $f_{\alpha_0}: X \to \mathbb{R}$ defined as

$$f_{\alpha_0}(x) = f(x, \alpha_0), \text{ for any } x \in X$$

is continuous w.r.t. $d_X$ (exercise). Because $D(\alpha_0)$ is nonempty and compact by assumption, Weierstrass theorem implies that

$$X^*(\alpha_0) := \arg \max_{x \in X} \{ f_{\alpha_0}(x) : x \in D(\alpha_0) \}$$

is nonempty. So we know that $X^*$ is nonempty-valued at $\alpha_0$.

**Step 2: $X^*$ is compact-valued at $\alpha_0$ and uhc at $\alpha_0$**

Let’s show this using Proposition 1.6.

Take any sequence $(\alpha_n)$ in $A$ convergent to $\alpha_0$, any sequence $(x_n)$ s.t. $x_n \in X^*(\alpha_n)$ for each $n \in \mathbb{N}$.

WTS: $\exists$ subsequence $(x_{n_k})$ convergent to some $x_0 \in X^*(\alpha_0)$.

Because $x_n \in X^*(\alpha_n) \subset D(\alpha_n)$ for each $n$, and because $D$ is compact valued at $\alpha_0$ and uhc at $\alpha_0$, by Proposition 1.6, $\exists$ subsequence $(x_{n_k})$ convergent to some $x_0 \in D(\alpha_0)$.

Take the $x_0$ found this way, and it is sufficient to show that $x_0 \in X^*(\alpha_0)$, i.e. $f(x_0, \alpha_0) \geq f(z, \alpha_0)$ for any $z \in D(\alpha_0)$.

Take any $z \in D(\alpha_0)$. WTS: $f(x_0, \alpha_0) \geq f(z, \alpha_0)$

Because $D$ is lhc at $\alpha_0$ and $\alpha_n \to \alpha_0$, by sequential definition of lhc (Proposition 1.15), there exists $K \in \mathbb{N}$ and $z_k \in D(\alpha_{n_k})$ for each $k > K$, s.t. $z_k \to z$.

Because $x_{n_k} \to x_0$, $\alpha_{n_k} \to \alpha_0$, we have $(x_{n_k}, \alpha_{n_k}) \to (x_0, \alpha_0)$ in $(X \times A, d_{X \times A})$. Because $f$ is continuous w.r.t. $d_{X \times A}$, we have $f(x_{n_k}, \alpha_{n_k}) \to f(x_0, \alpha_0)$.

Because $z_k \to x_0$, $\alpha_{n_k} \to \alpha_0$, we have $(z_k, \alpha_{n_k}) \to (x_0, \alpha_0)$ in $(X \times A, d_{X \times A})$. Because $f$ is continuous w.r.t. $d_{X \times A}$, we have $f(z_k, \alpha_{n_k}) \to f(z, \alpha_0)$.

For each $k$, we have $f(x_{n_k}, \alpha_{n_k}) \geq f(z_k, \alpha_{n_k})$ because $x_{n_k} \in X^*(\alpha_{n_k})$. Therefore we have $f(x_0, \alpha_0) \geq f(z, \alpha_0)$.

**Step 3: $f^*$ is continuous at $\alpha_0$**

By (1), we have $\alpha_0 \in \hat{A}$, i.e. $\alpha_0$ is in the domain of the function $f^*$. Therefore it makes sense to talk about continuity of $f^*$ at $\alpha_0$.

Let’s show the continuity of $f^*$ using the sequential definition of continuous functions.

Take any sequence $(\alpha_n)$ in $A$ convergent to $\alpha_0$. WTS: $f^*(\alpha_n) \to f^*(\alpha_0)$.

Suppose $f^*(\alpha_n) \to f^*(\alpha_0)$. Then there exist $\delta > 0$ s.t. for any $N \in \mathbb{N}$ there exists $\hat{n} > N$ s.t. $|f^*(\alpha_{\hat{n}}) - f^*(\alpha_0)| \geq \delta$. Then we can find a subsequence $(\alpha_{n_k})$ s.t. $|f^*(\alpha_{n_k}) - f^*(\alpha_0)| \geq \delta$ for each $k$.

For each $k$, because $\alpha_{n_k} \in \hat{A}$, the set $X^*(\alpha_{n_k})$ is nonempty. Arbitrarily take some $x_{k_l} \in X^*(\alpha_{n_k})$. Because $X^*$ is compact-valued at $\alpha_0$ and uhc at $\alpha_0$, by Proposition 1.6, there exists a subsequence $(x_{k_l})$ convergent to some point $x_0 \in X^*(\alpha_0)$. Because $\alpha_{n_k} \to \alpha_0$, $x_{k_l} \to x_0$, we have $(x_{k_l}, \alpha_{n_k}) \to (x_0, \alpha_0)$, and so $f(x_{k_l}, \alpha_{n_k}) \to f(x_0, \alpha_0)$. Because $x_{k_l} \in X^*(\alpha_{n_k})$, and $x_0 \in X^*(\alpha_0)$, we have $f(x_{k_l}, \alpha_{n_k}) = f^*(\alpha_{n_k})$ and $f(x_0, \alpha_0) = f^*(\alpha_0)$, and therefore

$$f^*(\alpha_{n_k}) \to f^*(\alpha_0)$$

However, we have $|f^*(\alpha_{n_k}) - f^*(\alpha_0)| \geq \delta$ for each $k$, by construction of the subsequence $(\alpha_{n_k})$. Contradiction.
By Theorem of Maximum, we can only conclude that the set of maximizers $X^*$ is uhc in the parameter $\alpha$. In fact, $X^*$ may easily fail to be lhc, even when the objective function $f$ and the constraint correspondence $D$ are continuous in the parameter $\alpha$. For example, consider the following problem:

$$\max_{(x_1, x_2) \in \mathbb{R}_+^2} \alpha x_1 + x_2 \text{ s.t. } p_1 x_1 + p_2 x_2 \leq m$$

where the parameters $\alpha > 0$, $p_1$, $p_2$, $m > 0$. Clearly, the objective function $f : \mathbb{R}_+^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ defined as

$$f(x, \alpha, p_1, p_2, m) := \alpha x_1 + x_2$$

is continuous. The constraint correspondence $D : \mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^2$ defined as

$$D(\alpha, p_1, p_2, m) := \left\{ x \in \mathbb{R}_+^2 : p_1 x_1 + p_2 x_2 \leq m \right\}$$

is nonempty- and compact-valued, and continuous at all $(\alpha, p_1, p_2, m) \in S$. Therefore the assumptions of the Theorem of Maximum are satisfied. However, it is not difficult to see that the set of maximizers $X^* : \mathbb{R}_+ \rightarrow \mathbb{R}_+^2$ is

$$X^*(\alpha, p_1, p_2, m) := \left\{ \begin{array}{ll}
(0, \frac{m}{p_2}) & , \text{if } p_1 > \alpha p_2 \\
(x \in \mathbb{R}_+^2 : p_1 x_1 + p_2 x_2 = m) & , \text{if } p_1 = \alpha p_2 \\
(\frac{m}{p_1}, 0) & , \text{if } 0 < p_1 < \alpha p_2
\end{array} \right.$$ 

which is clearly uhc but not lhc at the point $(\alpha, p_1, p_2, m)$ where $p_1 = \alpha p_2$.

If in addition to the assumptions in the Theorem 3.1, $D$ is convex-valued and $f$ is strictly concave in $x$, then $X^*$ is single-valued (see Proposition 2.3 in Lecture 5). That is, there is a unique maximizer for any $\alpha$ that satisfies the relevant conditions. In this case, we can think of $X^*$ as a continuous function $x^*$, such that $X^*(\alpha) = \{x^*(\alpha)\}$. Moreover, we have the following result if the parameter space $A$ and the space of the choice variable $X$ are both Euclidean spaces:

**Lemma 3.2.** Let $A$ be a set in $(\mathbb{R}^1, d_2)$, $X$ be a set in $(\mathbb{R}^m, d_2)$. Let $f : X \times A \rightarrow \mathbb{R}$ be a continuous function w.r.t. the metric $d_{X \times A}$ and $f(x, \alpha_0)$ is strictly quasi-concave in $x$ for some $\alpha_0 \in A$. Let $D : A \Rightarrow X$ be non-empty at $\alpha_0$, compact-valued at $\alpha_0$, convex-valued at $\alpha_0$, and continuous at $\alpha_0$. Then for any $\varepsilon > 0$, there exists $\delta_{\alpha_0} > 0$ (which might depend on $\alpha_0$) s.t. for $x \in D(\alpha_0)$,

$$|f^*(\alpha_0) - f(x, \alpha_0)| < \delta_{\alpha_0}$$

implies

$$d_2(x^*(\alpha_0), x) < \varepsilon.$$ 

If the conditions on $D$ and $f$ hold for $\forall \alpha \in A$, and $A$ is compact, then for any $\varepsilon > 0$, there exists $\delta > 0$ (independent of $\alpha$) s.t. for any $\alpha \in A$, $x \in D(\alpha)$,

$$|f^*(\alpha) - f(x, \alpha)| < \delta \Rightarrow d_2(x^*(\alpha), x) < \varepsilon.$$ 

This lemma says when we consider a maximization problem in a real space, if in addition to the conditions on the objective function $f$ and the constraint-set correspondence $D$ in Theorem 3.1, we also have $f$ is strictly quasi-concave in $x$, and $D$ is convex-valued, then for given $\alpha \in A$, as long as the value of the objective function evaluated at $x \in D(\alpha)$ is close enough to that evaluated at the maximizer $x^*(\alpha)$, then the point $x$ can get arbitrarily close to the maximizer.

Now we have the following theorem about the maximizers of a convergent sequence of parametrized functions on a parameterized constraint set.
Theorem 3.3. * Assume \( A, X, D \) satisfy the relevant conditions for \( \forall \alpha \in A \) as in the Lemma above. Let \( \{ f_n \} \) be a sequence of continuous functions \( f_n : X \times A \to \mathbb{R} \). Assume that for each \( n \) and \( \alpha \in A \), \( f_n(\cdot, \alpha) \) is strictly concave. Assume \( f : X \times A \to \mathbb{R} \) is also strictly concave in \( x \) and continuous. Let \( f_n \to f \) uniformly\(^5\). Let

\[
f_n^*(\alpha) = \max_{x \in D(\alpha)} f_n(x, \alpha), \quad n = 1, 2, \ldots
\]

\[
f^*(\alpha) = \max_{x \in D(\alpha)} f(x, \alpha)
\]

Then \( f_n^* \to f^* \) pointwise. If \( A \) is compact, then \( f_n^* \to f^* \) uniformly.

This theorem states that when we consider maximization problems in a real space, under certain conditions, the value function of objective function \( f_n \) under certain constraints (common to \( n \)) gets close to the value function of \( f \) under the same constraint, if \( (f_n) \) as a sequence of functions gets close to \( f \).

---

\(^5\) \( f_n \to f \) uniformly iff \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \) s.t. \( \sup_{\alpha \in A, x \in D(\alpha)} |f_n(x, \alpha) - f(x, \alpha)| < \varepsilon \). That is, uniform convergence of functions is convergence under the uniform metric defined for the space of functions: \( d(f, g) := \sup_{x \in D} |f(x) - g(x)| \).